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New realizations of observables in dynamical systems with second class constraints

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Abstract

In the Dirac bracket approach to dynamical systems with second class constraints observables are represented by elements of a quotient Dirac bracket algebra. We describe families of new realizations of this algebra through quotients of the original Poisson bracket algebra. Explicit expressions for generators and brackets of the algebras under consideration are found.

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1. Introduction

In a dynamical system with first class constraints, physical functions are elements of a Poisson bracket algebra of first class functions (see e.g. [1]). Observables are classes of the physical functions modulo the functions vanishing on the constraint surface.

In the Dirac bracket approach to a system with second class constraints [2] the original Poisson bracket is replaced by the Dirac one and constraints become first class. In this case all the functions on the phase space are first class and observables are elements of the Dirac bracket algebra of all the functions modulo the functions vanishing on the constraint surface.

The latter quotient algebra can be realized as a Poisson bracket algebra of the functions on the constraint surface [3]. Another useful realization can be obtained by using the Abelian conversion

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of second class constraints [4]. The algebra of observables is also realized as a quotient of the original Poisson algebra of first class functions [5].

Description of different realizations of observables is important in the context of deformation quantization [6]. One can expect quantization of a Poisson algebra not to depend on its realization. If this is the case, then using realizations of observables through the original Poisson bracket and Fedosov quantization [7], one can avoid deformation of the corresponding Dirac bracket algebra.

The aim of this article is to present the new Poisson bracket algebras which are isomorphic to the algebra of observables in dynamical systems with second class constraints.

The construction uses a set of nested subalgebras of the original Poisson bracket algebra of first class functions and their ideals which are generated by the functions vanishing on the constraint surface. The existence of such subalgebras imposes some restrictions on possible constraints. Solving the defining equations we find explicit expressions for generators of the algebras under consideration. This enables us to construct families of new isomorphic images of the algebra of observables. The new algebras are Poisson ones with respect to the original bracket.

The paper is organized as follows. In Section 2 we review a description of observables in systems with second class constraints. In Section 3 we describe the family of the constraints under consideration and find explicit expressions for the functions on phase space which serve as generators of new Poisson bracket algebras. These algebras are constructed and studied in Section 4. In Section 5 we describe new realizations of the algebra of observables in a system with second class constraints.

2. Realizations of observables

Let M be a phase space with the phase variables η^n, $n = 1 \ldots 2N$, and the Poisson bracket $[\eta^m, \eta^n] = \omega^{mn}(\eta)$. Let $H(\eta)$ be the original hamiltonian and $\varphi_j(\eta)$, $j = 1 \ldots 2J$, the second class constraints $\det[\varphi_j, \varphi_k]|_{\varphi=0} \neq 0$. We shall assume [2] that all the quantities vanishing on the constraint surface are linear functions of (φ_i).

The Hamilton equations of the system under consideration read

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta^n = [\eta^n, H_T], \qquad \varphi_j = 0. \tag{1}$$

Here

$$H_T = H + \lambda_j \varphi_j \tag{2}$$

and the $\lambda_j = \lambda_j(\eta)$ are defined by the equations

$$[H_T, \varphi_j]|_{\varphi=0} = 0. \tag{3}$$

From (1) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} f = [f, H_T], \qquad \varphi_j = 0 \tag{4}$$

for all $f = f(\eta)$.

Using (3) one can write Eq. (4) as

$$\frac{\mathrm{d}}{\mathrm{d}t} f = [f, H_T]_D, \qquad \varphi_j = 0. \tag{5}$$

Here the Dirac bracket was introduced:

$$[g, h]_D = [g, h] - [g, \varphi_j]c_{jk}[\varphi_k, h], \qquad c_{jk}[\varphi_k, \varphi_l] = \delta_{jl}.$$

The constraints (φ_j) are first class with respect to the Dirac bracket: $[\varphi_j, \varphi_k]_D = 0$ and the physical functions are defined by the equations

$$[f, \varphi_j]_D|_{\varphi=0} = 0$$

which are satisfied identically. Let A be the space of functions on M and $\Phi \subset A$ be the subspace of the functions which vanish on the constraint surface. Then the algebra of observables is the Dirac bracket algebra A/Φ. Note that A/Φ is also an algebra with respect to the pointwise multiplication and hence A/Φ is a Poisson algebra.

Let $\{f\} \in A/\Phi$ be the coset represented by $f \in A$. Then, using (5) and (3) one can obtain the Hamilton equations for observables:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{f\} = [\{f\}, \{H\}]_D. \tag{6}$$

In a recent article [5] a new approach to quantization of the system (1) was proposed. Let Υ be the algebra of the functions which are quadratic in φ_j and let Ω be the algebra of first class functions:

$$\Omega = \{f \in A \mid [f, \varphi_j]|_{\varphi=0} = 0\}. \tag{7}$$

For $u \in \Upsilon$ we have $u|_{\varphi=0} = 0$. The set Υ includes the element $\varphi_j\varphi_j$ and hence from the equations $u = 0, u \in \Upsilon$, it follows that $\varphi_j = 0$. Thus, the constraints (φ_j) and Υ are equivalent and we can replace Eq. (1) by

$$\frac{\mathrm{d}}{\mathrm{d}t}f = [f, H_T], \qquad u_{ij}\varphi_i\varphi_j = 0. \tag{8}$$

Here $u_{ij} = u_{ij}(\eta)$ are arbitrary functions. In contrast with the original ones, the new constraints Υ are first class.

In this approach the algebra of physical functions consists of all the functions which satisfy the equations

$$[f, u] \in \Upsilon \tag{9}$$

for all $u \in \Upsilon$. One can show that these equations are equivalent to the definition of first class functions (7) in the original second class system. Due to Eqs. (9) and (7) the algebra of observables is the Poisson algebra Ω/Υ.

Let $\{f\}^\bullet \in \Omega/\Upsilon$ be the coset represented by $f \in \Omega$. Then the equations for observables read

$$\frac{\mathrm{d}}{\mathrm{d}t}\{f\}^\bullet = [\{f\}^\bullet, \{H_T\}^\bullet]. \tag{10}$$

The present approach and the Dirac bracket one are related by the isomorphism of the algebras of observables Ω/Υ and A/Φ [5]:

$$T(\{g\}^\bullet) = \{g\}.$$

Below we shall obtain new realizations of the algebra A/Φ through quotients of the original Poisson algebra.

3. Generators of Ω_{s+1}

Let Ω_{s+1}, $s \in N$, be the space of the functions on M which are defined by the equations

$$[\varphi_j, \tilde{g}] \in \Upsilon_s. \tag{11}$$

Here

$$\Upsilon_s = \{u \in A \mid u = u_{j_1 \dots j_s} \varphi_{j_1} \dots \varphi_{j_s}, u_{j_1 \dots j_s}(\eta) \in A\}.$$

It is seen that $\Upsilon_{s+1} \subset \Omega_{s+1} \subset \Omega_s$, $\Omega_2 = \Omega$, $\Upsilon_{s+1} \subset \Upsilon_s$, $\Upsilon_2 = \Upsilon$ and $\Upsilon_1 = \Phi$. We shall write $\Upsilon_0 = \Omega_1 = A$.

To describe elements of Ω_{s+1} explicitly let us consider Eq. (11) with the boundary condition

$$\tilde{g}(\eta) \in \{g(\eta)\}. \tag{12}$$

From (12) it follows that $\tilde{g}_{|\varphi=0} = g$ or $\tilde{g} = g + v_i \varphi_i$ for some functions $v_i(\eta)$. Hence a solution to Eqs. (11) and (12) can be represented in the form

$$\tilde{g} = g + \sum_{r=1}^{s} \frac{1}{r!} v_{i_1 \dots i_r}(\eta) \varphi_{i_1} \dots \varphi_{i_r} + v_{i_1 \dots i_{s+1}}(\eta) \varphi_{i_1} \dots \varphi_{i_{s+1}}. \tag{13}$$

Note that the last term of (13) satisfies (11) for arbitrary $v_{i_1 \dots i_{s+1}}$.

We shall assume that $v_{i_1 \dots i_r}$, $r = 1 \dots s$, is symmetric:

$$v_{\underbrace{\dots i_a \dots i_b \dots}_{r}} - v_{\underbrace{\dots i_b \dots i_a \dots}_{r}} \in \Upsilon_{p+1-r}, \quad p \geq s. \tag{14}$$

Substituting (13) into (11) and using (14) we get

$$[\varphi_j, g] + v_{i_1}[\varphi_j, \varphi_{i_1}] + \sum_{r=2}^{s} \frac{1}{(r-1)!} \left([\varphi_j, v_{i_1 \dots i_{r-1}}] + v_{i_1 \dots i_r}[\varphi_j, \varphi_{i_r}] \right) \varphi_{i_1} \dots \varphi_{i_{r-1}} \in \Upsilon_s.$$

It is easy to see that a solution to these equations is

$$v_{i_1 \dots i_r} = (-1)^r D_{i_r} \dots D_{i_1} g, \qquad r = 1 \dots s. \tag{15}$$

Here $D_i = c_{ij}[\varphi_j, \cdot]$.

One can check that the D_i satisfy the commutator relations

$$D_i D_j - D_j D_i = [c_{ij}, \cdot]_D \tag{16}$$

and for $u \in \Upsilon_r$

$$D_i u \in \Upsilon_{r-1}. \tag{17}$$

Now let us consider Eq. (14). It is sufficient to find a solution to these equations for $a = k+1$, $b = k$, $k = 1 \dots r-1$. Substituting (15) into (14) and using (16) we have

$$D_{i_r} \dots D_{i_{k+2}} [c_{i_{k+1} i_k}, D_{i_{k-1}} \dots D_{i_1} g]_D \in \Upsilon_{p+1-r}, \qquad k = 1 \dots r-1. \tag{18}$$

A solution to these equations is given by

$$c_{ij} = a_{ij}^{-1} + v_{ij}, \qquad v_{ij}(\eta) \in \Upsilon_{p-1}. \tag{19}$$

Here the a_{ij} are constant and $\det(a_{ij}) \neq 0$.

To check that c_{ij} satisfy Eq. (18) observe that for $f \in A$

$$[c_{ij}, f]_D \in \Upsilon_{p-1}$$

and due to (17)

$$D_{i_r} \dots D_{i_{k+2}}[c_{i_{k+1}i_k}, D_{i_{k-1}} \dots D_{i_1} g]_D \in \Upsilon_{p+k-r} \subset \Upsilon_{p+1-r}$$

for all $k = 1 \dots r-1$. Thus for c_{ij} (19) expressions (13) and (15) give us a solution to Eq. (11) with the boundary condition (12).

Let now $\tilde{g}'$ be another solution to Eq. (11) with the same boundary condition $\tilde{g}' \in \{g\}$. Then $\sigma = \tilde{g} - \tilde{g}'$ is a solution to (11)

$$[\varphi_j, \sigma] \in \Upsilon_s \tag{20}$$

and $\sigma = \sigma_i \varphi_i$ for some $\sigma_i = \sigma_i(\eta)$.

From (20) it follows that

$$[\varphi_{j_1}, \dots, [\varphi_{j_{m-1}}, [\varphi_j, \sigma]]] \in \Upsilon_{s-m+1}. \tag{21}$$

Assume that

$$\sigma = \sigma_{i_1 \dots i_m}(\eta)\varphi_{i_1} \dots \varphi_{i_m}. \tag{22}$$

Substituting (22) into (21) for $m \leq s$ we get $\sigma_{i_1 \dots i_m}|_{\varphi=0} = 0$ and hence $\sigma = \sigma_{i_1 \dots i_{m+1}}(\eta)\varphi_{i_1} \dots \varphi_{i_{m+1}}$.

For $m = s$,

$$\sigma = \sigma_{i_1 \dots i_{s+1}}(\eta)\varphi_{i_1} \dots \varphi_{i_{s+1}}.$$

We have proved the proposition:

Proposition 3.1. *For c_{ij} (19) and $g \in A$ the set $\{g\} \cap \Omega_{s+1}$, $s = 1 \dots p$, consists of all the expressions*

$$\tilde{g} = g + \sum_{r=1}^{s} \frac{(-1)^r}{r!} \left(D_{i_r} \dots D_{i_1} g\right) \varphi_{i_1} \dots \varphi_{i_r} + \nu_{i_1 \dots i_{s+1}} \varphi_{i_1} \dots \varphi_{i_{s+1}}, \tag{23}$$

where $\nu_{i_1 \dots i_{s+1}}(\eta)$ are arbitrary functions.

In what follows we shall assume that c_{ij} is given by (19) and $1 \leq s \leq p$. From condition (19) it follows that

$$[\varphi_i, \varphi_j] = a_{ij} + w_{ij}, \quad w_{ij}(\eta) \in \Upsilon_{p-1}. \tag{24}$$

It is convenient to introduce the notation

$$L_s(g) = g + \sum_{r=1}^{s} \frac{(-1)^r}{r!} \left(D_{i_r} \dots D_{i_1} g\right) \varphi_{i_1} \dots \varphi_{i_r}.$$

The hamiltonian in Ω_{s+1} is

$$\tilde{H}_T = L_s(H) + u, \qquad u \in \Upsilon_{s+1}. \tag{25}$$

It can be represented in the form (2), satisfies Eq. (3) and hence belongs to the family of admissible hamiltonians.

4. Algebraic properties of Ω_{s+1}

Proposition 4.1. *Ω_{s+1} is an algebra and Υ_{s+1} is an ideal of Ω_{s+1} with respect to the original Poisson bracket, Dirac bracket and pointwise multiplication.*

The proof is straightforward.

Due to this proposition Ω_{s+1}, Υ_{s+1} and $\Omega_{s+1}/\Upsilon_{s+1}$ are Poisson algebras with respect to $[\cdot,\cdot]_D$ as well as $[\cdot,\cdot]$.

Let

$$\tilde{g}_a = L_s(g_a) + u_a, \qquad u_a \in \Upsilon_{s+1}, \tag{26}$$

$a = 1, 2$, be some elements of Ω_{s+1} and let $\{\tilde{g}_a\}_s \in \Omega_{s+1}/\Upsilon_{s+1}$ be the coset represented by $\tilde{g}_a \in \Omega_{s+1}$.

Proposition 4.2. *For $\tilde{g}_1, \tilde{g}_2$ (26) one has*

$$\begin{aligned} &[\tilde{g}_1, \tilde{g}_2] = L_s([g_1, g_2]_D) + \tilde{u}_{12}, \quad [\tilde{g}_1, \tilde{g}_2]_D = L_s([g_1, g_2]_D) + \tilde{v}_{12}, \\ &\tilde{g}_1\tilde{g}_2 = L_s(g_1 g_2) + \tilde{w}_{12}, \qquad \tilde{u}_{12}, \tilde{v}_{12}, \tilde{w}_{12}, \in \Upsilon_{s+1}. \end{aligned} \tag{27}$$

Proof. One can check that $[\tilde{g}_1, \tilde{g}_2]$ satisfies Eq. (11) with the boundary condition $[\tilde{g}_1, \tilde{g}_2] \in \{[g_1, g_2]_D\}$. Due to results of the previous section one has

$$[\tilde{g}_1, \tilde{g}_2] = L_s([g_1, g_2]_D) + \tilde{u}_{12}, \qquad \tilde{u}_{12} \in \Upsilon_{s+1}.$$

Other statements of the proposition are proved by using similar arguments. □

Corollary 4.3. *The Dirac bracket algebra $\Omega_{s+1}/\Upsilon_{s+1}$ is isomorphic to the algebra $\Omega_{s+1}/\Upsilon_{s+1}$ with respect to the original Poisson bracket.*

Proof. From Eq. (27) we have

$$[\{\tilde{g}_1\}_s, \{\tilde{g}_2\}_s] = [\{\tilde{g}_1\}_s, \{\tilde{g}_2\}_s]_D = \{L_s([g_1, g_2]_D)\}_s. \quad \square$$

5. New realizations of observables

Theorem 5.1. (i) *The Dirac bracket algebra A/Φ is isomorphic to the algebra $\Omega_{s+1}/\Upsilon_{s+1}$ with respect to the original Poisson bracket.*

(ii) *A/Φ and $\Omega_{s+1}/\Upsilon_{s+1}$ are isomorphic with respect to the pointwise multiplication.*

Proof. Let us define the linear function $T_s : \Omega_{s+1}/\Upsilon_{s+1} \to A/\Phi$

$$T_s(\{g\}_s) = \{g\}.$$

Each function $g' \in \{g\} \bigcap \Omega_{s+1}$ can be written in the form (23). Hence the inverse function $T_s^{-1} : A/\Phi \to \Omega_{s+1}/\Upsilon_{s+1}$ is given by

$$T_s^{-1}(\{g\}) = \{L_s(g)\}_s.$$

Computations show that T_s is the homomorphism

$$T_s([\{g\}_s, \{f\}_s]) = [T_s(\{g\}_s), T_s(\{f\}_s)]_D$$

and hence A/Φ and $\Omega_{s+1}/\Upsilon_{s+1}$ are isomorphic.

To prove the second statement we observe that T_s is the homomorphism with respect to pointwise multiplication:

$$T_s(\{g\}_s\{f\}_s) = T_s(\{gf\}_s) = \{gf\} = \{g\}\{f\} = T_s(\{g\}_s)T_s(\{f\}_s). \quad \square$$

Corollary 5.1. *$\Omega_{s+1}/\Upsilon_{s+1}, s = 1 \dots p$, are isomorphic to each other as Poisson algebras.*

The function $T_{s+k,s}$ which defines isomorphism between $\Omega_{s+k+1}/\Upsilon_{s+k+1}, k \geq 0$, and $\Omega_{s+1}/\Upsilon_{s+1}$ is given by

$$T_{s+k,s}(\{g\}_{s+k}) = \{g\}_s.$$

Theorem 5.1 gives us new realizations of the algebra of observables A/Φ through the original Poisson bracket. For a given system we have p realizations, where p is defined by the form of c_{ij} (19) or $[\varphi_i, \varphi_j]$ (24).

For $p = 1$ $[\varphi_i, \varphi_j] \in \Upsilon_0 = A$ and there is only one realization $\Omega_2/\Upsilon_2 = \Omega/\Upsilon$.

Let us consider the constraints of a gauge theory $(\pi_\alpha(\eta), \chi_\beta(\eta)), \alpha, \beta = 1 \dots M$,

$$[\pi_\alpha, \pi_\beta] = f_{\alpha\beta\gamma}(\eta)\pi_\gamma, \quad [\pi_\alpha, \chi_\beta] = g_{\alpha\beta}(\eta), \quad [\chi_\alpha, \chi_\beta] = 0. \tag{28}$$

Here π_α are first class constraints, χ_β are gauge fixing conditions and $\det(g_{\alpha\beta}) \neq 0$. With the new gauge fixing conditions $\chi'_\alpha = g^{-1}_{\beta\alpha}\chi_\beta$ Eq. (28) take the form (24) for $p = 2$:

$$[\pi_\alpha, \pi_\beta] = f_{\alpha\beta\gamma}(\eta)\pi_\gamma, \quad [\pi_\alpha, \chi'_\beta] = \delta_{\alpha\beta} + a_\alpha(\eta)\chi'_\alpha, \quad [\chi'_\alpha, \chi'_\beta] = b_\alpha(\eta)\chi'_\alpha.$$

Here $a_\alpha(\eta)$ and $b_\alpha(\eta)$ are some functions. In this case the observables can be realized by Ω/Υ or Ω_3/Υ_3.

When $[\varphi_i, \varphi_j] = \psi_{ij}(\varphi)$, where $\psi_{ij}(\varphi)$ are functions of the constraints (φ_j) only, there is an infinite series of such realizations. The number p can be used for classification of second class constraints.

According to (4) and (3) the Hamilton equation in $\Omega_{s+1}/\Upsilon_{s+1}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\{f\}_s = [\{f\}_s, \{\tilde{H}_T\}_s].$$

Here $\tilde{H}_T$ is given by (25).

6. Conclusion

In the present article we have obtained new realizations of observables in dynamical systems with second class constraints. The observables are realized as Poisson algebras with respect to the original bracket. We have found the restrictions which are imposed on constraints by construction of such algebras. The number of possible realizations of the observables for a given system can be used for classification of second class constraints. We have obtained explicit expressions for generators and brackets of all the algebras under consideration.

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References

[1] D.M. Gitman, I.V. Tyutin, Quantization of Fields with Constraints, Springer-Verlag, Berlin, 1990.
[2] P.A.M. Dirac, Lectures on Quantum Mechanics, Yeshiva University, New York, 1964.
[3] J. Sniatycki, Dirac brackets in geometric dynamics, Ann. Inst. H. Poincare 20 (1974) 365–372.
[4] I. Batalin, I. Tyutin, Existence theorem for the effective gauge algebra in the generalized canonical formalism with abelian conversion of second class constraints, Int. J. Mod. Phys. A 6 (1991) 3255–3282.
[5] A.V. Bratchikov, Realization of Dirac bracket algebras through first class functions and quantization of constrained systems, Lett. Math. Phys. 61 (2002) 107–111.
[6] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerovich, D. Sternheimer, Deformation theory of quantization. I. Deformation of symplectic structures, Ann. Phys. 111 (1) (1978) 61–110.
[7] B. Fedosov, A simple geometric construction of deformation quantization, J. Differential Geom. 40 (2) (1994) 213–238.